# STOCHASTIC EQUATIONS AND EVOLUTION
# FAMILIES IN THE SPACE OF FORMAL MAPPINGS

I.YA. SPECTORSKY

NATIONAL TECHNICAL UNIVERSITY OF UKRAINE (KPI), KIEV, UKRAINE

## 1. INTRODUCTION

Let's give some consideration leading to the notion of formal mapping.
Let's consider a stochastic equation

$$y(t) = y(0) + \int\limits_0^t a(\tau)(y(\tau))d\tau + \int\limits_0^t b(\tau)(y(\tau))dw(\tau), \qquad 0 \leqslant t \leqslant T$$

in Hilbert space $Y$ (all Hilbert spaces are supposed to be real and separable). Here $w$ is a Wiener process, associated with canonical triple $H_+ \subset H_0 \subset H_-$, with Hilbert-Schmidt embeddings, $a$ and $b$ are continuous mappings from $[0,T] \times Y$ into spaces $Y$ and $\mathcal{L}_2(Y)$ respectively, $y$ is unknown random process taking its values in space $Y$, $y(0)$ is nonrandom initial condition.

Let coefficients $a$ and $b$ be analytical functions with respect to $y \in Y$ under fixed $t$, and $a(t)(0) = 0$, $b(t)(0) = 0$. I.e., for all $y \in Y$ the following expansions into power series holds:

$$a(t,y) = \sum_{k \geqslant 1} a_k(t)(y, y, \ldots, y), \quad b(t,y) = \sum_{k \geqslant 1} b_k(t)(y, y, \ldots, y).$$

Here $a_k(t)$ and $b_k(t)$ are $k$-linear continuous operators from $Y$ to $Y$ and from $Y$ to $\mathcal{L}_2(Y)$ respectively.

Let this equation have the unique solution $y(t) = S(t)(y)$, and the following expansion holds: $S(t)(y) = \sum_{k \geqslant 1} S_k(t)(y, y, \ldots, y)$. In this case, function $a(t) \circ S(t)(y)$ and $b(t) \circ S(t)(y)$ can be expanded into power series by $y$, with the expansion coefficients to be calculated by the following formulas:

$$(a \circ S)_n = \sum_{k=1}^n \sum_{j_1 + j_2 + \cdots + j_k = n} a_k(S_{j_1}, \ldots, S_{j_k})$$

$$(b \circ S)_n = \sum_{k=1}^n \sum_{j_1 + j_2 + \cdots + j_k = n} b_k(S_{j_1}, \ldots, S_{j_k}).$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Substituting expansions for $a(y)$ and $b(y)$ into original equation and comparing corresponding coefficients, one can obtain the system of linear stochastic equations for $n$-linear continuous operators $S_n$, $n \geqslant 1$:

$$
\begin{cases}
S_1(t) & = S_1(0) + \int\limits_0^t a_1(\tau)S_1(\tau)d\tau + \int\limits_0^t b_1(\tau)(S_1(\tau), dw(\tau)), \\[2mm]
S_2(t) & = \int\limits_0^t a_1(\tau)S_2(\tau)d\tau + \int\limits_0^t b_1(\tau)(S_2(\tau), dw(\tau)) + \\[2mm]
& \quad + \int\limits_0^t a_2(\tau)(S_1(\tau), S_1(\tau))d\tau + \int\limits_0^t b_2(\tau)(S_1(\tau), S_1(\tau))dw(\tau), \\[2mm]
& \vdots \\[2mm]
S_n(t) & = \int\limits_0^t \sum_{k=1}^n \sum_{j_1+j_2+\cdots+j_k=n} a_k(\tau)(S_{j_1}(\tau), S_{j_2}(\tau), \ldots, S_{j_k}(\tau))d\tau + \\[2mm]
& \quad + \int\limits_0^t \sum_{k=1}^n \sum_{j_1+j_2+\cdots+j_k=n} b_k(\tau)(S_{j_1}(\tau), S_{j_2}(\tau), \ldots, S_{j_k}(\tau))dw(\tau), \\[2mm]
& \vdots
\end{cases}
\tag{1}
$$

It's easy to see that the first $n$ equations of system (1) with any $n$ are closed with respect to $S_k$, $1 \leqslant k \leqslant n$. It gives us a possibility to solve the system recursively: find $S_1$ from the first equation, find $S_2$ from the second one, using $S_1$ just found, find $S_3$ from the third equations, using $S_1$ and $S_2$ already found, so on.

Let's note that system (1) remains valid in the case, when $a_k$ and $b_k$ ($k \geqslant 1$) are not coefficients of expansion of analytical function into power series: all sums contained in (1) are finite, and we may implement recursion procedure without any worrying about series convergence. These considerations lead to the notion of formal mapping.

## 2. FORMAL MAPPINGS

Let $Y$ and $Z$ be Hilbert spaces.

**Definition 1.** A sequence $a = (a_k)_{k \geqslant 1}$, where $a_k$ ($k \geqslant 1$) are $k$-linear continuous mappings from $Y$ into $Z$, we call formal mapping from $Y$ into $Z$. Denote by $\mathcal{L}_\infty(Y, Z)$ a space of formal mappings from $Y$ into $Z$.

For formal mappings $a \in \mathcal{L}_\infty(X, Y)$ and $b \in \mathcal{L}_\infty(Y, Z)$ composition operation is introduced: $b \circ a \in \mathcal{L}_\infty(X, Z)$, $(b \circ a)_n = \sum_{k=1}^n \sum_{j_1+\cdots+j_k=n} b_k(a_{j_1}, a_{j_2}, \ldots, a_{j_k})$.

Formal mapping $a \in \mathcal{L}_\infty(Y, Y)$, denoted by $\mathrm{Id}_Y$, is called identical one if $a_1 = \mathrm{id}_Y$ and $a_n = 0$ with $n \geqslant 2$.

**Example.** Any analytical function $a$ can be associated with formal mapping $a$, whose coefficients $a_k$ are coefficients of expanding $a$ into power series: $a(y) = \sum_{k \geqslant 1} a_k(y, y, \ldots, y)$. In this case, composition of analytical functions $c(y) = b(a(y))$ corresponds to composition of formal mappings $c = b \circ a$, and identical function $a(y) = y$ corresponds to identical formal mapping $\mathrm{Id}_Y$.

To find more about formal mappings see [2].

2

## 3. STOCHASTIC EQUATIONS IN THE SPACE OF FORMAL MAPPINGS

Let's consider stochastic equation in the space of formal mappings:

$$S(t,s) = S(s,s) + \int_s^t a(\tau)(S(\tau,s))d\tau + \int_s^t b(\tau)(S(\tau,s))dw(\tau), \qquad (2)$$

where $a$ and $b$ are continuous mappings from $[0,T]$ into spaces $\mathcal{L}_\infty(Y,Y)$ and $\mathcal{L}_\infty(Y,\mathcal{L}_2(Y,H_0))$ respectively, $S(t,s)$ is unknown random process taking its values in space $\mathcal{L}_\infty(Y,Y)$, $S(s,s)$ is initial condition, measurable with respect to $\sigma$-algebra $\mathfrak{F}_s = \sigma(w(\tau), 0 \leqslant \tau \leqslant s)$. Equation (2) is considered component-wise, i.e. (2) is equivalent to the following system:

$$
\begin{cases}
S_1(t,s) = S_1(s,s) + \int_s^t a_1(\tau)S_1(\tau,s)d\tau + \int_s^t b_1(\tau)(S_1(\tau,s),dw(\tau)), \\
n \geqslant 2: \quad S_n(t,s) = S_n(s,s) + \\
\qquad + \int_s^t \sum_{k=1}^n \sum_{j_1+j_2+\cdots+j_k=n} a_k(\tau)(S_{j_1}(\tau,s),S_{j_2}(\tau,s),\ldots,S_{j_k}(\tau,s))d\tau + \\
\qquad + \int_s^t \sum_{k=1}^n \sum_{j_1+j_2+\cdots+j_k=n} b_k(\tau)(S_{j_1}(\tau,s),S_{j_2}(\tau,s),\ldots,S_{j_k}(\tau,s))dw(\tau).
\end{cases}
$$
$$(3)$$

**Theorem 1.** *Let $a_n$ and $b_n$ ($n \geqslant 1$) be functions, continuous with respect to $t \in [0,T]$, taking their values in spaces $\mathcal{L}_2(Y^{\otimes n}, Y)$ and $\mathcal{L}_2(Y^{\otimes n} \otimes H_0, Y)$ respectively. Additionally we suppose that the initial conditions satisfy the following requirements:*

$$S_1(s,s) - \mathrm{id}_Y \in \mathcal{L}_2(Y,Y); \qquad \forall\, n \geqslant 2: S_n(s,s) \in \mathcal{L}_2(Y^{\otimes n}, Y).$$

*Then system (2) has a solution $S(t,s)$, unique within stochastic equivalence, such that:*

$$S_1(t,s) - \mathrm{id}_Y \in \mathcal{L}_2(Y,Y); \qquad \forall\, n \geqslant 2: S_n(t,s) \in \mathcal{L}_2(Y^{\otimes n}, Y).$$

**Theorem 2.** *Let the condition of Theorem 1 hold, and operators $S(t,s)$ ($0 \leqslant s \leqslant t \leqslant T$) be solutions to equation (2) with initial condition $S(s,s) = \mathrm{Id}_Y$.*
*Then operator family $\{S(t,s)\}_{0 \leqslant s \leqslant t \leqslant T}$ is evolution one, i.e.:*

$$S(s,s) = \mathrm{Id}_Y, \quad S(t,\tau) \circ S(\tau,s) = S(t,s) \text{ with } s \leqslant \tau \leqslant t.$$

Further we suppose that formal mapping $S(t) = S(t,0)$ is solution to (2) with initial condition $S(0) = \mathrm{Id}_Y$. Let's rewrite system (3) for process $S(t)$:

$$
\begin{cases}
S_1(t) = \mathrm{id}_Y + \int_0^t a_1(\tau)S_1(\tau)d\tau + \int_0^t b_1(\tau)(S_1(\tau,s),dw(\tau)), \\
S_n(t) = \int_0^t a_1(\tau)S_n(\tau)d\tau + \int_0^t b_1(\tau)(S_n(\tau),dw(\tau)) + \\
\qquad + \int_0^t f_n(\tau)d\tau + \int_0^t g_n(\tau)dw(\tau), \quad n \geqslant 2,
\end{cases}
$$

where $f_n(t) = \sum\limits_{k=2}^{n} \sum\limits_{j_1+j_2+\cdots+j_k=n} a_k(t)(S_{j_1}(t), S_{j_2}(t), \ldots, S_{j_k}(t)),$

$$g_n(t) = \sum_{k=2}^{n} \sum_{j_1+j_2+\cdots+j_k=n} b_k(t)(S_{j_1}(t), S_{j_2}(t), \ldots, S_{j_k}(t)).$$

Inasmuch as $f_n$ and $g_n$ contains only $S_k$ with $k \leqslant n - 1$, this system can be solved recursively, calculating $S_1$, $S_2$, ..., $S_n$, ... . To give recursion procedure in a more convenient way, we can use one explicit formula for solution to linear nonhomogeneous stochastic equation (see [2]):

$$S_n(t) = \int_0^t S_1(t,\tau)(f_n(\tau))d\tau + \int_s^t S_1(t,\tau)(g_n(\tau), \widehat{dw(\tau)}),$$

where $S_1(t, \tau)$ is evolution operator, satisfying linear equation for $S_1$, and symbol $\widehat{dw(\tau)}$ denotes extended stochastic integral, treated as adjoint to stochastic derivative operator.

## References

1. Daletsky Yu. L., *Algebra of compositions and nonlinear equations*, Algebraic and Geometric Methods in Mathematical Physics, (1996), Kluwer Academic Publisher, Printed in Netherlands, 277-291.
2. Spectorsky I. Ya., *Explicit formula for solution of linear nonhomogeneous stochastic equation (in Russian)*, Deponed in State Science Technical Library of Ukraine 02-01-1996, 424 - UK 96.

I.Ya. Spectorsky, postgraduate student, National Technical University of Ukraine "Kiev Polytechnic Institute", Chair of Mathematical Methods of System Analysis, pr. Pobedy, 37, Kiev, Ukraine.